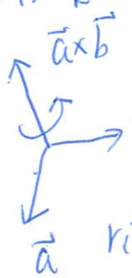
Cross Product

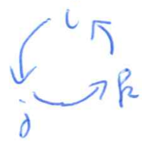
Let $\vec{a}, \vec{b} \in \mathbb{R}^3$. Define

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\hat{i} + (-a_1 b_3 + a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



right-hand rule



Properties

① $i \times j = k$, $j \times k = i$, $k \times i = j$,

② $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$,

③ $\vec{a} \times \vec{a} = 0$,

④ $\vec{a} \times (\alpha \vec{b} + \beta \vec{c}) = \alpha\, \vec{a} \times \vec{b} + \beta\, \vec{a} \times \vec{c}$

⑤ $(\alpha \vec{a}) \times \vec{b} = \vec{a} \times (\alpha \vec{b}) = \alpha (\vec{a} \times \vec{b})$.

Dot product & cross product

~ $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$ (a direct check)

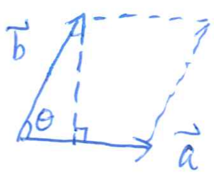
~ $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$, θ angle bet. $\vec{a}$ & $\vec{b}$.

(Pf: $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - |a|^2 |b|^2 \cos^2\theta$
$= |\vec{a}|^2 |b|^2 \sin^2\theta$.)

~ $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$

(a direct check)

~ $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$ (Jacobi's identity)



geometric meaning

~ $|\vec{a} \times \vec{b}|$ is the area of the parallelogram formed by $\vec{a}, \vec{b}$

~ $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ is the volume of the parallelepiped formed by $\vec{a}, \vec{b}, \vec{c}$.

(draw it yourself)